

Significance of solutions of the inverse Biot-Savart problem in thick superconductors

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Abstract. The evaluation of current distributions in thick superconductors from field profiles near the sample surface is investigated theoretically. A simple model of a cylindrical sample, in which only circular currents are flowing, reduces the inversion to a linear least squares problem, which is analyzed by singular value decomposition. Without additional assumptions about the current distribution (e.g. constant current over the sample thickness), the condition of the problem is very bad, leading to unrealistic results. However, any additional assumption strongly influences the solution and thus renders the solutions again questionable. These difficulties are unfortunately inherent to the inverse Biot-Savart problem in thick superconductors and cannot be avoided by any models or algorithms.

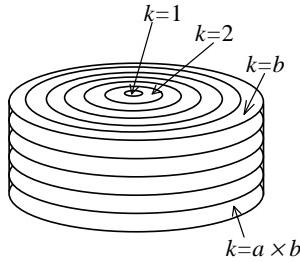


Figure 1. Radial and axial division of the mathematical sample into toroids.

1. Introduction

The calculation of two dimensional current distributions from the magnetic field generated by them was successfully performed with various algorithms [1, 2, 3, 4, 5, 6, 7]. All these algorithms are based on the assumption, that the current flows in infinitely thin layers [1, 3, 6] or that the current is uniform [5, 7] (or averaged [2]) over the sample thickness. In most cases, only the z-component of the magnetic field (orthogonal to the sample surface) is measured on a 2D grid. From this discrete field distribution the two components (x, y) of the current are calculated on a similar grid within the sample by matrix inversion [3, 4, 5] or by fast Fourier transformation [1, 6, 7]. It is straightforward to extend these algorithms to thick superconductors, if the current can be assumed to be constant over the sample thickness [8, 9]. Unfortunately, this condition is never fulfilled in real superconductors due to material inhomogeneities and due to the self field. The influence of the inhomogeneities can easily be seen by the differences between the remnant field profiles taken at the top and the bottom surface of thick bulk samples [10]. Although the effect of the self field cannot be detected directly, it is obvious that the changing magnitude (largest in central layers) and direction of the self field results in a z-dependence of the current density. Nevertheless, solutions obtained under these wrong assumptions are used for the analysis of spatial variations of the sample properties. The aim of this paper is to point out that it is in principle not possible to calculate the current distribution within a thick superconductor from the z-component of the remnant field profile, if a realistic experimental error is taken into account. It will also be shown, that any unjustified assumptions lead to artifacts in the resulting solutions. The influence of the experimental error, of noise reduction and of additional (wrong) assumptions is demonstrated for a simplified model system and the resulting solutions are compared with the true (known) current distribution.

2. Model System

The sample is chosen to be a cylinder with radius R and thickness d . This sample is divided a times axially (perpendicular to the sample axis), resulting in a layers. Each layer is divided again b times radially (into hollow cylinders), leading to $n := a \times b$ toroids of rectangular cross section (Fig. 1). The toroids are labeled by k , starting with

the innermost toroid of the first layer. The numeration continues in the first layer until $k = b$, then jumps to the innermost toroid of the second layer and so on. Currents are assumed to flow circularly with a constant current density in each toroid. The magnetic induction at a point \vec{r} , generated by each toroid, is then proportional to the actual current density in the toroid. This leads to a system of linear equations between the current densities in each toroid j_k and the z-component of the magnetic induction at m discrete points $B_z(\vec{r}_l)$ (the data points B_l):

$$B_l = \sum_{k=1}^n M_{l,k} j_k \quad l = 1, \dots, m \quad (1)$$

or more compactly

$$\vec{B} = M \vec{j}. \quad (2)$$

The m -dimensional vector \vec{B} and the n -dimensional vector \vec{j} are representing discrete values of B_z at m points outside the sample and of the current densities in the n toroids, respectively. The coefficients $M_{l,k}$ are calculated by integrating the Biot-Savart law over the volume of the corresponding toroid. In the following it is assumed that the number of data points m exceeds the number of toroids n , i.e. the equation is overdetermined, which is favorable for the suppression of experimental noise.

This model system was chosen because of its simplicity. It is radially symmetric, which reduces the number of data points significantly. At the same spatial resolution the field profile is determined by m instead of $4m^2$ data points, e.g., for a grid width of 0.5 mm , m is 29 for a field profile from $r = 0$ to $r = 14 \text{ mm}$ instead of $4m^2 = 3364$ for a two dimensional quadratic grid. A similar reduction is obtained for the grid inside the sample, but in this case the knowledge of the current direction leads to an additional reduction of the number of unknown parameters by a factor of three (magnitude of the current density instead of its three components) and current conservation is fulfilled automatically. The resulting systems of equations are relatively small and can be solved numerically without any additional algorithmic problems, which would be inherent to large systems of equations. The condition of this simplified problem is certainly better than in a more complex system. It is possible with this model to study the inverse Biot-Savart problem without any numerical problems and it is obvious that a more realistic model behaves worse. The model restricts the possible solutions to circular currents, which can also be present in real samples. Therefore, any algorithm for the inversion of more general current distributions should also be able to invert circular currents. Thus, the results obtained from this model system are also valid for any other algorithmic implementation of the inverse Biot-Savart problem in thick superconductors.

3. Condition Number

A first insight into the problem can be obtained by calculating the condition number K of the coefficient Matrix M [11]. The condition number reflects the error propagation:

$$\frac{\Delta B}{B} \leq K \frac{\Delta j}{j} \quad (3)$$

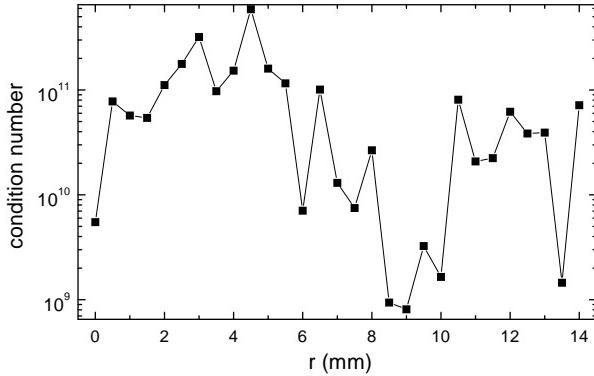


Figure 2. Condition number in the case of an axial field profile as a function of the radial coordinate.

The relative error of the calculated current density can be K times larger than the relative experimental error of the magnetic field values. In principle, significant results can only be obtained, if the product of the condition number and the relative experimental error is much smaller than one. Since equality is the worst case scenario of the inequality 3, reasonable results can usually be obtained as long as this product is not much larger than unity, but in this case a careful analysis of the error propagation is needed.

The condition number is reasonably small, if the sample is divided only radially ($a=1$). Assuming a sample of radius $R = 12.5\text{ mm}$ with a height of $d = 500\text{ nm}$ and inverting a radial flux profile with 29 points between $r = 0$ and $r = 14\text{ mm}$, measured 500 nm above the sample surface, the condition number is 5.7 for ten radial divisions ($b=10$). This small thickness and the small gap between the sample and the field profile is representative for magneto-optical measurements on thin films, where inversion schemes are well established [6, 7]. This quite small condition number (K cannot be smaller than one) becomes 16.3, if the sample thickness is enhanced to $d = 10\text{ mm}$, which is typical for melt textured bulk samples. If a gap of $\Delta z = 0.2\text{ mm}$ is assumed (representative for Hall probe measurements) K is further increased to 27.3. These values are still suitable for a proper inversion. Therefore, numerically stable [8, 9] (but not necessarily correct !) solutions can be obtained from the inversion of the Biot-Savart law, if the current is assumed to be homogeneous along the sample thickness. The situation immediately changes without this restriction. If the sample is divided only axially ($a=10$, $b=1$), the condition number for the same number of free parameters and for the same geometry becomes 4.7×10^8 . One cannot expect to get correct results in this case for any reasonable experimental error bars. In order to exclude the possibility, that this awful condition is a consequence of the *radial* flux profile for the calculation of an *axial* current distribution, the condition number was calculated for axial field profiles (29 points from $\Delta z = 0.2$ to $\Delta z = 28.2\text{ mm}$) at different radial positions (Fig. 2). The condition number depends on the radial position of the axial field profile, but does not become smaller than 8.1×10^8 . Providing a radial field profile at each Δz (841 data

points) reduces K only slightly to 1.4×10^8 . It is obviously impossible to calculate the current densities of only 10 layers from 841 data points (with realistic experimental errors). This result indicates immediately that a general inversion of the Biot-Savart law is simply not possible for thick superconductors.

4. Single Value Decomposition (SVD)

The single value decomposition (SVD) is a powerful tool for the solution of systems of linear equations [12]. It not only solves the equations in a numerically stable form, but also gives insights into the problems, which may arise. The $m \times n$ matrix M of Eqn. 2 can be decomposed into three matrices: $M = U\Lambda V^T$. Since the $n \times n$ matrix V is orthonormal, its columns \vec{v}_i can be interpreted as basis vectors for the representation of the current densities. Λ is a $n \times n$ diagonal matrix. The (positive) diagonal elements are called singular values λ_i , with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. While it is generally possible that some singular values become zero, this case will be excluded in the following, since it does not occur in the actual model systems. The columns \vec{u}_i of the $m \times n$ matrix U are pairwise orthonormal and are basis vectors for the representation of \vec{B} . Since m was assumed to be larger than n the basis vectors \vec{u}_i only span a subspace of \vec{B} , the range of M . M maps each basis vector \vec{v}_i to \vec{u}_i multiplied by the corresponding singular value λ_i :

$$M\vec{v}_i = \lambda_i \vec{u}_i. \quad (4)$$

Problems occur, if some singular values are much smaller than others. Current components corresponding to such small singular values only add very little to the field profile and are very hard or impossible to detect, especially if their contribution becomes smaller than the experimental error. To be more concrete, the singular values were calculated for a sample, which is representative for melt textured bulk samples ($R = 12.5 \text{ mm}$, $d = 10 \text{ mm}$). It was divided into 100 toroids ($a = b = 10$) and 824 data points were assumed to be available at 29 radial flux profiles with 29 points each ($\Delta z = 0.2 \dots 28.2 \text{ mm}$, $r = 0 \dots 14 \text{ mm}$). The singular values are plotted in Fig. 3, normalized by the largest singular value λ_1 , since their absolute values depend only on the actual units. They vary by twelve orders of magnitude, which explains the bad condition of the problem. If the experimental error is about 1 %, only the first 12 current components \vec{v}_i are expected to generate a signal that is larger than this experimental error. In principle, this is only true, if the magnitude of the current components with large singular values is not much smaller than the magnitude of the other current components, but this condition is fulfilled for any reasonable current distribution. Since most of the current components are not generating a significant signal, they cannot be determined. On the other hand, one can at least calculate the current distribution within the sample projected to this 12 dimensional subspace. The image of this subspace can be defined as the "experimental range" of the matrix M .

Although an experimental error of 1 % seems to be high for the measurement of the magnetic field, the total expected error for a typical experiment is of that order of

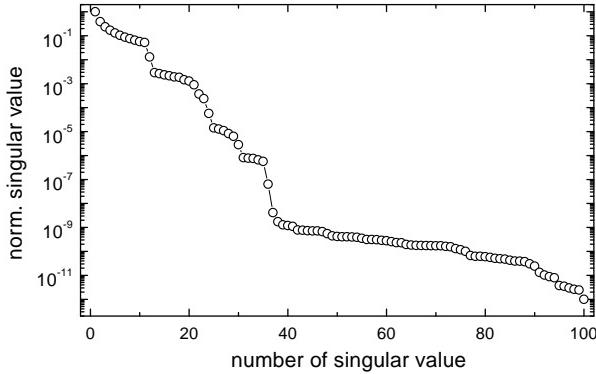


Figure 3. Singular values of the model system.

magnitude, since other (in principle not stochastic) sources of error have to be added. The gap between the Hall probe or the magneto-optical layer is usually not known exactly, relaxation of the currents decreases the field during the measurement, oxygen pick-up in liquid nitrogen can change its temperature and so on. A decrease of the experimental error leads to an increase of the dimension of the experimental range, but the basic problem remains the same.

The best approximating solution (in the least square sense) can be obtained easily after the single value decomposition:

$$\vec{j} = V \Lambda^{-1} U^T \vec{B} \quad (5)$$

The multiplication with U^T projects \vec{B} to the range of M and performs a basis transformation (new basis: u_i). All experimental errors, which are out of the range of M , are mapped to zero and do not induce any error in the calculated currents at all. The coefficients (in the new representation) of the remaining \vec{B} are then divided by the corresponding singular values λ_i . Finally, V performs another basis transformation. Components of the measured magnetic induction pointing in the direction \vec{u}_i are amplified by $1/\lambda_i$. Especially components of the experimental error corresponding to a small singular value (large i) will be strongly amplified. From the following worst case scenario, the condition number of the system can be easily calculated. The current distribution \vec{j} within the sample is just proportional to \vec{v}_1 generating a field $\vec{B} = |\vec{j}| \lambda_1 \vec{u}_1$. An experimental error $\Delta \vec{B}$ proportional to \vec{u}_n induces an error in the derived current distribution $\Delta \vec{j} = |\Delta \vec{B}| / \lambda_n \vec{v}_n$. This leads immediately to the inequation 3 with $K = \frac{\lambda_n}{\lambda_1}$. In the system under consideration the condition number is about 10^{12} (Fig. 3). In order to illustrate the influence of this awful condition number, the field profile \vec{B}_0 generated by a totally homogeneous current distribution of one (in arbitrary units) was calculated, i.e. the same current density is assumed to flow in each toroid. To check for numerical problems, \vec{B}_0 was inverted with relation 5. The maximum deviation from unity due to round-off errors was as small as 0.14 %, i.e. there are no numerical problems. Each component of \vec{B}_0 was then multiplied by a random number between 0.99 and 1.01, which simulates a (stochastic) experimental error of 1 %. The inversion leads to a completely different current distribution (Fig. 4) with huge (positive and

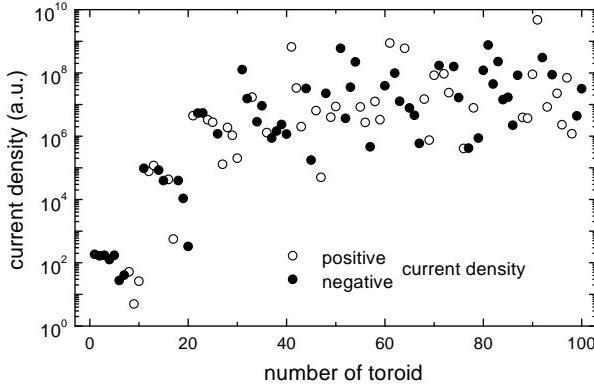


Figure 4. Best approximating current distribution for data with typical experimental error.

negative) current densities. The largest value is 4.7×10^9 , about half of the upper limit for the error propagation $K\Delta B = 10^{10}$. With the aid of the single value decomposition, not only the condition number of the whole problem can be derived, but the condition number of each component of the field (coefficients of u_i) or of the corresponding current density (v_i) can be obtained separately. As pointed out above, only the first twelve components are significant (if an experimental error of 1 % is assumed), i.e. they are expected to generate a signal that is larger than the experimental error. Inverting only these twelve significant field components of \vec{B}_0 (or equivalently, projecting the whole solution to the 12 dimensional subspace of \vec{j}) leads to a current distribution, which is far from being constant and is in average, much smaller than one (0.73). In Fig. 5a, data points between two grid lines represent the radial current distribution within one layer. This obviously wrong current distribution fits the data perfectly, even better than the assumed error of 1 % in the Euclidean norm. In this norm the deviation is only 4.5×10^{-4} , in the maximum norm about 10^{-3} . Calculating the relative deviation in each point separately, it is found to be always smaller than 3.5×10^{-3} (in 11 points out of 841 this cannot be done, because the value is numerically zero there). This is a general behavior of ill conditioned problems. The solution nicely agrees with the data, as long as it is correct in the significant subspace. In the remaining space the solution can be assumed more or less arbitrarily. Therefore, agreement with the inverted data, does not indicate the correctness of the solution. If the noisy version of \vec{B}_0 is inverted in that way, the "solution" looks quite similar, with deviations of up to 14 %, significantly lower than the theoretical upper limit of the error propagation (100 %). This justifies the rule of thumb that the experimental error times the condition number should not be much larger than one. The projection to the significant subspace represents some sort of filtering. Any filtering (e.g., disregarding the highest frequency of the Fourier transformation) not only reduces the noise, but also changes the solution, because it cannot distinguish between experimental noise and the "true" signal. Its influence on the solution is the stronger the worse the condition of the problem and one should be very careful with solutions obtained after filtering. In the present case the decrease of

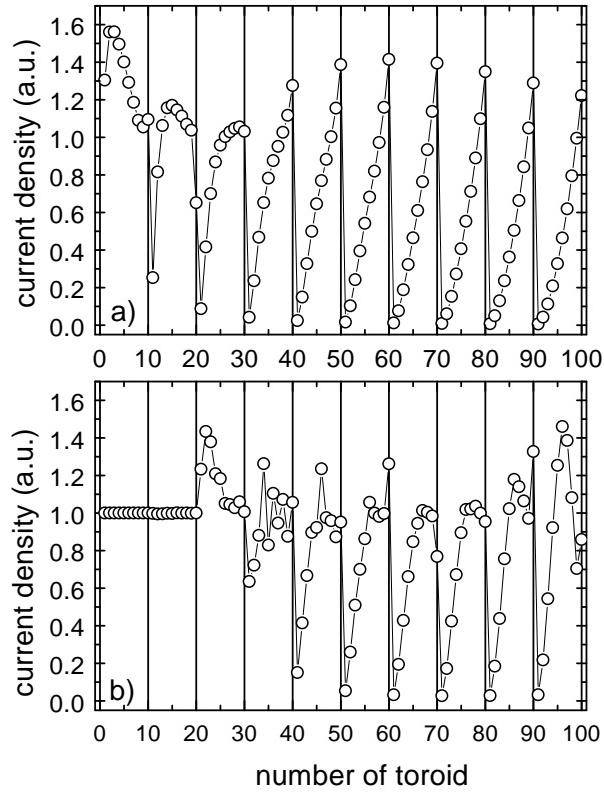


Figure 5. Inversion of the "experimental range" for an assumed experimental error of (a) 10^{-2} and (b) 10^{-9} .

the currents from the top to the bottom could be interpreted as a deterioration of the sample properties with increasing distance from the seed, the higher currents at the sample edges as the influence of the self field. Both explanations are expected and, therefore, plausible, but *cannot* be derived from the present data, since the field profile was calculated assuming a constant current density.

Figure 5b shows the projection of the exact solution to the first 50 v_i s, which would require the experimental error to be smaller than 10^{-9} . Even in this unrealistic case the correct solution is obtained only in the uppermost two layers, the situation in the bottom layers does not improve significantly. Although it might be possible to improve the condition of this specific problem by a few orders of magnitude by optimizing the grid and the sample division, this would not be sufficient, even if the experimental error could be reduced by one or two orders of magnitude. In a more realistic system, i.e. allowing the currents to flow in arbitrary directions, the condition is expected to be even worse.

5. Homogeneous Current along the Sample Thickness

As already pointed out, one possibility to obtain a reasonable condition for the inverse Biot-Savart problem is to assume the currents to be homogeneous along the sample thickness [8, 9]. The influence of a violation of this condition is discussed in this section.

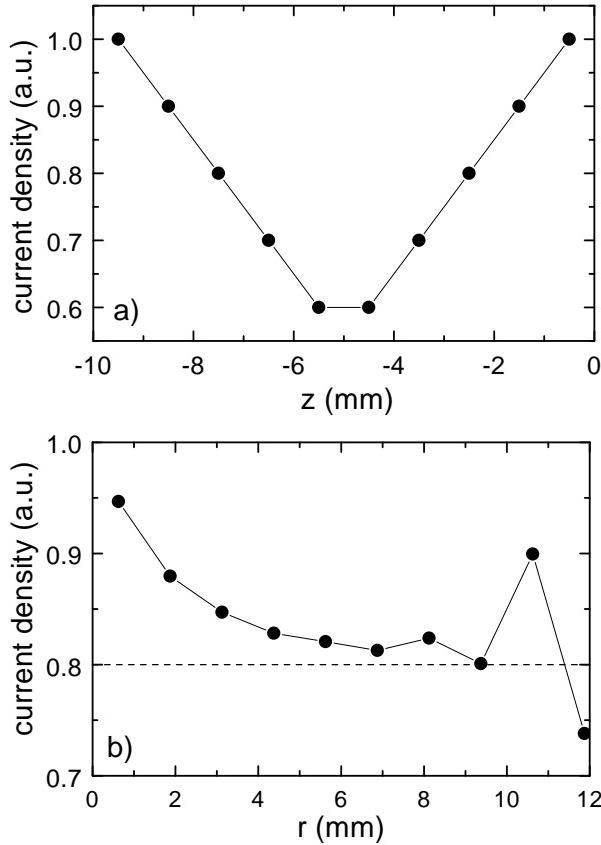


Figure 6. (a) Assumed axial current distribution and (b) calculated radial current distribution. The correct average current is constant 0.8 (broken line).

The field profile of a sample ($R = 12.5\text{ mm}$, $d = 10\text{ mm}$) was calculated for a radially constant but axially changing current density ($a = 10$, $b = 1$). The model current distribution is plotted in figure 6a, $z = 0$ and $z = -10\text{ mm}$ correspond to the top and bottom surface, respectively. Such a behavior is expected even in a totally homogeneous sample due to the self field, which is largest in the central layers. Since the critical current density decreases with field (at least at low fields), the currents are largest at the surface layers. The calculated field profile was then inverted (wrongly) assuming the current to be constant over the sample thickness ($a = 1$, $b = 10$). The resulting radial current distribution (Fig. 6b) is larger than the correct value of 0.8 averaged over z (except near the sample edge) and a pronounced radial dependence is observed. The assumption of homogeneous current along the sample thickness, which is never fulfilled in thick superconductors, is not appropriate for the inverse Biot-Savart problem in these samples, since neither the correct magnitude nor the correct radial dependence of the current are obtained.

6. Conclusions

It was shown that it is generally not possible to derive the three dimensional current distribution within a thick superconductor from a three dimensional distribution of the z-component of the magnetic induction, which is generated by this current distribution. Under additional assumptions or by filtering, reasonably looking but wrong current distributions are obtained. The field generated by these wrong current distributions agrees with the correct field within experimental accuracy. Agreement between measured data and calculated current densities is, therefore, no indication for the correctness of the solution or of any of the underlying assumptions. Such solutions are most probably wrong and may lead to wrong conclusions.

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